Research statement

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My research lies in the fields of Differential Geometry, Geometric Analysis and Partial Differential Equations. In particular, I am interested in the analysis and applications of geometric flows such as Ricci flow, mean curvature flow. My PhD dissertation and some of my subsequent work has focused in the understanding of the Ricci flow on manifolds with boundary. Moreover, my most recent research interests also include issues on the singularity formation of Type I Ricci flows, such as the size of their singular set, and also on the Ricci flow with non-smooth initial data. In the following sections I will describe my work, and also outline my plans for future research.

1. The Ricci flow on manifolds with boundary.

1.1. **Background.** The Ricci flow equation on a closed manifold M

$$\partial_t g = -2\operatorname{Ric}(g) \tag{1.1}$$

was introduced by Hamilton in [Ham82] and shown to have a unique solution for a short time given any smooth initial Riemannian metric g^0, so that $g(0) = g^0$. The diffeomorphism invariance of the Ricci tensor makes the equation only weakly parabolic and for that reason Hamilton had to implement Nash's implicit function theorem to prove local existence.

However, DeTurck in [DeT83] showed that the Ricci flow equation, given a fixed background metric $\tilde{g}$, is equivalent to the following strongly parabolic system

$$\partial_t \bar{g} = -2\operatorname{Ric}(\bar{g}) + \mathcal{L}_{\mathcal{W}}\bar{g}, \tag{1.2}$$

where $\mathcal{W} = \bar{g}^{pq}(\Gamma(\bar{g})^r_{pq} - \tilde{\Gamma}(\tilde{g})^r_{pq})$. By standard parabolic theory this system has a unique solution for a short time and it turns out that if ϕ_t are the diffeomorphisms generated by $-\mathcal{W}$ satisfying $\phi_0 = \mathrm{id}_M$, then $g = \phi^*\bar{g}$ solves (1.1).

Although Ricci flow has been extensively studied and proven to be a powerful tool in Geometric Analysis, there is still not a well developed theory for it on manifolds with boundary. Even local existence, finding the right boundary conditions that guarantee existence and uniqueness of the flow, had been known only under very special assumptions on the boundary. In particular, the obvious Dirichlet problem, specifying the induced metric at the boundary, turns out not to satisfy the required parabolicity condition.

Regarding a Neumann-type problem, Shen in [She96] supplements (1.2) with parabolic boundary conditions which allow him to prove a short time existence result for the Ricci flow in which the second fundamental form is controlled. However, this result requires the initial metric to have umbilic boundary (i.e. its second fundamental form is a constant in space multiple of the metric). Later on, Pulemotov in [Pul12] obtains local existence of the Ricci flow in the class of metrics which have boundary of constant mean curvature, controlling the mean curvature of the boundary along the flow. In both results, the restrictions of umbilicity, and constant mean curvature are due to the fact that $\mathcal{W}$ is not required to vanish along ∂M and is instead tangent to the boundary. Also, note that in [Pul12] solutions are not going to be unique, as will become clear below.

1.2. **Short-time existence and uniqueness results.** In this section I describe results on the local existence and uniqueness of solutions to several boundary value problems for the Ricci flow.

1.2.1. *A general boundary value problem for the Ricci flow.* In the following I will describe a boundary value problem for (1.2) which is inspired by the work of Anderson [And08] on boundary value problems for Einstein metrics. In this work, he shows that the conformal class of the induced metric on the boundary and its mean curvature provide an elliptic boundary value problem for the Einstein equations.

To introduce some notation, if g is a smooth Riemannian metric on M we will denote by $\mathcal{H}(g)$ the mean curvature of the boundary and by u^T the part of a tensor u, tangential to the boundary. Moreover, if γ is a Riemannian metric on ∂M, $[\gamma]$ will denote its conformal class, namely

$$[\gamma] = \left\{\gamma' = \phi^2\gamma \text{ , for all positive functions } \phi \text{ on } \partial M\right\}.$$

Let g^0 be an arbitrary smooth Riemannian metric on M, $\gamma(x,t) \in C^\infty(\partial M \times [0,+\infty))$ a smooth time-dependent family of metrics on ∂M and a function $\eta(x,t) \in C^\infty(\partial M \times [0,+\infty))$. We assume that they satisfy the zeroth order compatibility conditions

$$\mathcal{H}(g^0) = \eta|_{t=0}, \quad \left[(g^0)^T\right] = [\gamma|_{t=0}]. \tag{1.3}$$

Then, the following theorem is proven in [Gia12].

Theorem 1. *Let g^0, γ, η as above. There exists a $T > 0$ and a smooth family of metrics $g(t)$ for $0 < t \leq T$ that solves (1.1) and satisfies on ∂M the boundary conditions*

$$\mathcal{H}(g(t)) = \eta(t), \quad \left[g^T(t)\right] = [\gamma(t)], \text{ for all } t \in (0,T]. \tag{1.4}$$

In addition, $g(t)$ converges in the Cheeger-Gromov $C^{1,\alpha}$ sense (i.e. up to diffeomorphisms that fix the boundary) to g^0 and C^∞ away from the boundary, as $t \to 0$.

Moreover, if g^0, γ, η satisfy necessary higher order compatibility conditions then the regularity of the flow at $t = 0$ can be improved and $g(t)$ converges to g^0 in the Cheeger-Gromov $C^{k+2,\alpha}$ sense.

An estimate of the minimal time of existence of the flow holds in terms of the geometry of the initial metric, and appropriate norms of the boundary data. (see Theorem 2 in [Gia12]*).*

Remark 1.1. The boundary conditions (1.4) suffice to determine uniquely a Ricci flow with given initial data, assuming some additional regularity. An important consequence of this is that an isometry of the initial metric which leaves $[\gamma], \eta$ invariant remains an isometry along flow.

Remark 1.2. The simplest possible boundary data, Dirichlet data (prescribing the induced metric on the boundary), don't provide a well posed problem for the Ricci-DeTurck equation or even the Einstein equations (see [And08]). Thus, the results above on uniqueness and preservation of isometries are not expected to hold if one only has control of the induced metric on the boundary, and not of its mean curvature.

1.2.2. *Static flow and Bartnik's data.* In General Relativity, the simplest possible spacetime metrics are the so called static metrics. These are metrics on $N^4 = \mathbb{R} \times M$ of the form

$$-V^2dt^2 + g_M,$$

where V is a smooth positive function and g_M a Riemannian metric on M. If the metric on N is Ricci flat then the triplet (M, g, V) is called a static vacuum spacetime. Such solutions yield Ricci flat warped product Riemannian metrics on $S^1 \times M$ of the form

$$h = V^2d\theta^2 + g_M. \tag{1.5}$$

Moreover, the static vacuum equations become an elliptic system on M:

$$\operatorname{Ric}(g) = V^{-1}\operatorname{Hess}_g V \tag{1.6}$$
$$\Delta_g V = 0. \tag{1.7}$$

List, in [Lis08] initiated a flow approach towards understanding these, physically motivated metrics. He introduced the following geometric flow for Riemannian metrics g and functions u on M whose fixed points correspond to static vaccum solutions, under a conformal change.

$$\partial_t g = -2\operatorname{Ric}(g) + 2\alpha_n du \otimes du \tag{1.8}$$
$$\partial_t u = \Delta_g u. \tag{1.9}$$

Later, Gulcev, Oliynyk and Woolgar in [GOW10] studied the long-time existence of this flow assuming rotational symmetry.

On the other hand, considering the Ricci flow of the warped product metrics on $S^1 \times M$ of the form (1.5) we obtain a different flow with the following evolution equations for g and V

$$\partial_t g = -2\operatorname{Ric}(g) + 2V^{-1}\operatorname{Hess}_g V \tag{1.10}$$
$$\partial_t V = \Delta_g V. \tag{1.11}$$

A normalized version of this flow was introduced by Shi and Hu in [HS12] in the asymptotically hyperbolic setting.

Generalizing a result of Lichnerowicz, Anderson in [And99] proves that a complete static vacuum three-dimensional space is flat and V is constant. Thus, non-trivial solutions of (1.6)-(1.7) are incomplete, hence the need for a better understanding of boundary value problems for this system. This is also related to a definition of quasi-local mass and a conjecture of Bartnik in [Bar89] and [Bar02]:

Conjecture. Given a Riemannian metric g in the unit 3-ball $B(0,1)$, there is a unique static asymptotically flat Riemannian metric $\bar{g}$ in $\mathbb{R}^3 \setminus B(0,1)$ which satisfies

$$\bar{g}^T = g^T,$$
$$\mathcal{H}(\bar{g}, \partial B(0,1)) = \mathcal{H}(g, \partial B(0,1)),$$

where $\mathcal{H}$, g^T denote the mean curvature and the induced metric respectively of the boundary ∂B^3.

There has been a lot of interesting work on this conjecture from the point of view of elliptic PDE, by Miao in [Mia03], Anderson and Khuri in [AK11] and Anderson in [And13b, And13a]. However, a flow approach may provide a different insight to the problem.

Motivated by this it is natural to ask whether the flow (1.10)-(1.11) has solutions for short time, under Bartnik's boundary conditions, namely prescibing the induced metric g^T and mean curvature $\mathcal{H}(g)$ of ∂M. Surprisingly, although (1.6)-(1.7) with Bartnik's data form an elliptic boundary value problem, they do not yield a parabolic boundary value problem for (1.10)-(1.11).

However, a slight modification of Bartnik's data are parabolic, and it is possible to prove the following theorem using the techniques in [Gia12]. Below M is a compact manifold with boundary, the theorem though remains valid for non-compact M under additional control of the geometry at infinity.

Theorem 2. *Let g_0 be a Riemannian metric and V_0 a smooth positive function on M, and $\gamma(t)$, $\eta(t)$ arbitrary smooth one parameter families of metrics and functions on ∂M (satisfying zero order* *compatibility conditions of similar to (1.3)). Then, there exists a smooth solution* $g(t), V(t)$ *to (1.10)-(1.11), with* $0 < t \leq T$ *for small* $T > 0$*, satisfying*

$$\begin{aligned} g^T(t) &= \gamma(t), \\ \mathcal{H}(g(t)) + V^{-1}N(V) &= \eta(t), \end{aligned}$$

for every t*. Moreover,* $g(t), V(t)$ *converge to* g_0, V_0 *in* $C^{1,\alpha}$*, up to diffeomorphisms of* M *that fix* ∂M*. Also, this convergence is smooth away from* ∂M *and is improved up to the boundary if higher order compatibility conditions hold among* $(g_0, V_0, \gamma(t), \eta(t))$*.*

In collaboration with Marcus Khuri, we study the possibility of generalizing Bartnik's boundary conditions for the stationary vacuum problem and its corresponding flow. Moreover, although Bartnik's data are merely elliptic and not parabolic, we would like to understand the problem further, since the boundary conditions in Theorem 2 are less natural from the point of view of Bartnik's conjecure.

1.3. **A continuation principle for the Ricci flow.** As Ricci flow is a nonlinear system of equations, its solutions are not expected to exist for all time. For that reason, a central issue in the study of Ricci flow is understanding what is the nature of the singularities that appear and also what kind of conditions suffice for the continuation of the flow. If the manifold is closed, it is a well known result of Hamilton [Ham82] that the flow exists as long as the norm of the curvature tensor stays bounded. The following theorem generalizes this when the manifold has a boundary, and is a combination of results in [Gia12, Gia13].

Theorem 3. *Let* $g(t)$ *be a smooth Ricci flow on a compact manifold with boundary* M *with maximal time of existence* $T < \infty$*. Suppose that there exist smooth boundary data* $([\gamma], \eta)$ *defined for all* $t \in [0, T')$*,* $T' > T$ *such that* $[g^T(t)] = [\gamma(t)]$ *and* $\mathcal{H}(g(t)) = \eta$ *for all* $t \in [0, T)$*. Then*

$$\sup_{0 \leq t < T} \left(\sup_{x \in M} |\operatorname{Rm}(g(t))|_{g(t)} + \sup_{x \in \partial M} |\mathcal{A}(g(t))|_{g(t)} \right) = +\infty,$$

where $\mathcal{A}(g(t))$ *is the second fundamental form of* ∂M *with respect to the metric* $g(t)$*. Moreover, when* $\dim M = 3$ *we can conclude that*

$$\sup_{M \times [0,T)} |\operatorname{Rm}(g(t)|_{g(t)} = +\infty.$$

The improvement in dimension three is due to a result which bounds the second fundamental form of the boundary in terms of the ambient Riemannian curvature, the boundary injectivity radius and the boundary conditions. Its proof is via a blow up argument, where a rescaled sequence of counterexamples converges to a flat manifold with minimal boundary which is conformally equivalent to the plane. To obtain a contradiction we use Liouville's theorem for bounded subharmonic functions, hence the restriction on the dimension on the statement of the theorem. It would be interesting to investigate what happens in higher dimensions. Do singularities of the Ricci flow in which only the second fundamental form blows up while the curvature remains bounded exist?

1.4. **Boundary estimates for the Ricci flow.** In the study of the singularities of the Ricci flow a strategy is to consider a sequence of appropriate rescalings of the flow as it approaches a singularity. In this process a compactness theorem for sequences of Ricci flows is essential. Such result usually exploits the smoothing character of Ricci flow to assert convergence in the smooth topology, assuming merely bounds on the curvature and the injectivity radius.

In particular, Shi in [Shi89] (see also [Ham95b]) shows that along a Ricci flow in some finite time interval $[0,T]$, the derivatives of the curvature Rm in any interval $[\tau,T]$, $\tau>0$ are uniformly controlled in terms of the curvature. Such estimates are local in nature and there is no requirement the Ricci flow to be complete. However, on manifolds with boundary they are not enough to control to higher order the geometry *near* the boundary.

On the other hand, the blow-up analysis of singularities that develop on the boundary would require a version of these estimates to hold near the boundary. Moreover, a compactness theorem for sequences of Ricci flows on manifolds with boundary would also require higher order control of the second fundamental form $\mathcal{A}$ of the boundary.

In the following I will describe some progress on this problem, which appears in [Gia13]. A Ricci flow with Λ-controlled boundary is, roughly speaking, a Ricci flow with a choice $\gamma(t)$ of representatives of $[g^T(t)]$ with the property that, in γ-harmonic coordinates, $\gamma(t)$ and the mean curvature $\mathcal{H}(g(t))$ are controlled in the Hölder sense of order $m+\epsilon$ and $m-1+\epsilon$ respectively, where m is a large integer.

First, the following theorem is the analogue of the global high derivative of curvature estimates.

Theorem 4. *Let $(M,g(t),\gamma(t))$ be a complete Ricci flow with Λ-controlled boundary, with $t\in[0,T]$. Suppose*

(1) $|\operatorname{Rm}(g(t))|_{g(t)}\leq K$ *in* M *and* $|\mathcal{A}(g(t))|_{g^T(t)}\leq K$ *on* ∂M *for all* t.
(2) $i_{b,g(0)}\geq i_0$.

For any $j=1,\dots,m-2$ and $\tau>0$, there exists a constant $C=C(n,\tau,T,\Lambda,l,j,K,i_0)>0$ such that for all $t\in[\tau,T]$

$$|\nabla^j\operatorname{Rm}(g(t))|_{g(t)}\leq C\quad\text{in } M \text{ and}\quad |\nabla^{j+1}\mathcal{A}(g(t))|_{g^T(t)}\leq C\quad\text{on }\partial M.$$

Here, $i_{b,g(0)}$ the “boundary injectivity radius”, namely the size of the collar neighbourhood of ∂M diffeomorphic to $\partial M\times[0,i_{b,g(0)})$ via the normal exponential map of the boundary.

We remark that these estimates can be localized, in the sense that a similar result holds in the case $(M,g(t))$ are not complete metric spaces.

Once these higher order estimates have been established, it is possible to prove the following generalization of Hamilton's compactness theorem [Ham95a], where the limit is a manifold with boundary.

Theorem 5. *Let (M_k,p_k) be a pointed sequence of compact manifolds with boundary, $p_k\in\partial M_k$, and $(g_k(t),\gamma_k(t))$ be complete Ricci flows on M_k, $t\in(a,b]$ with Λ-controlled boundary. Assume $0\in(a,b]$ and*

(1) $|\operatorname{Rm}(g_k)|_{g_k}\leq K$ *in* $M_k\times(a,b]$ *and* $|\mathcal{A}(g_k)|_{g_k^T}\leq K$ *in* $\partial M_k\times(a,b]$.
(2) $i_{b,g_k(0)}\geq i_0$.

for all k. Then there is a pointed manifold with boundary (M_∞,p_∞), a complete Ricci flow $g_\infty(t)$ on M_∞ and a family of metrics $\gamma_\infty(t)$ on ∂M_∞ such that, up to subsequence,

$$(M_k,g_k(t),\gamma_k(t),p_k)\to(M_\infty,g_\infty(t),\gamma_\infty(t),p_\infty).$$

The order of convergence up to the boundary, depends on the order m of control of the boundary data.

Here, the notation $(M_k,g_k(t),\gamma_k(t),p_k)\to(M_\infty,g_\infty(t),\gamma_\infty(t),p_\infty)$ means that $g_k(t)$ and $\gamma_k(t)$ converge to $g_\infty(t)$ and $\gamma_\infty(t)$ respectively, after pulling back by appropriate diffeomorphisms (the same for both sequences) .

1.5. **Directions for future research.** In the following I summarize some of the questions/problems that I am working on at the moment or plan to work on in the near future. I will begin with more specific, concrete problems and move towards larger and more difficult issues.

1.5.1. *Stability questions for the Ricci flow on manifolds with boundary.* In the near future I plan to investigate issues related to the stability and convergence of the Ricci flow on manifolds with boundary.

On manifolds without boundary there has already been a large amount of interesting work on the topic. On closed manifolds for instance, stabililty around Ricci flat metrics has been studied by Günther, Isenberg and Knopf in [GIK02], Sesum in [Ses06b], Haslhofer in [Has12] and Haslhofer-Müller in [HM13]. In the non-compact setting, Woolgar and Oliynyk in [OW07] show that the Ricci flow on rotationally symmetric asymptotically flat manifolds exists for all time and converges in the pointed Cheeger-Gromov sense to the Euclidean space, provided the initial data don't contain minimal hyperspheres. In a more general case, Schnürer, Schulze and Simon in [SSS08] and [SSS11] study the stability of Ricci flow around Euclidean and hyperbolic space respectively, without any symmetry assumption. Also, Bamler in [Bam10a, Bam10b] studies the stability of hyperbolic manifolds with cusps and symmetric spaces, adapting techniques from the work of Koch and Lamm in [KL12].

On manifolds with boundary, one could consider flat or Ricci flat manifolds with boundary (M, g^0) and aim to understand the long time behaviour of the Ricci flow, or the static flow, given initial data near g^0 in an appropriate sense. Alternatively, one could consider g^0 as initial condition and construct a non-trivial flow by perturbing the boundary conditions. It would be interesting to discover situations in which the flow converges, provided the boundary conditions stabilize.

In particular, some of the questions I would like to explore are the following. Let $M = \mathbb{R}^3 \setminus B(0,1)$.

(1) Let g_0 be an asymptotically flat metric close to the Euclidean metric on M inducing mean curvature and conformal class on ∂M close to that of the Euclidean metric. Is there a Ricci flow, with the mean curvature and conformal class of the boundary fixed in time, converging to a flat metric as $t \to \infty$?
(2) Let the initial data be the Euclidean metric on M, and prescribe the mean curvature and/or the conformal class (as a functions of time) to stabilize to different values as $t \to \infty$. Is it the case that there is a Ricci flow converging to a different flat metric on M?

1.5.2. *Aspects of the general theory of the Ricci flow on manifolds with boundary.* Below I will describe some fundamental issues that arise in the study of the Ricci flow on manifolds with boundary, and outline some directions for future investigation.

Perelman in [Per02], [Per03b], [Per03a] made a breakthrough in Ricci flow introducing remarkable, deep ideas, leading to the proof of Thurston's Geometrization Conjecture and the Poincare Conjecture. I would like to investigate how, and of course if, Perelman's ideas extend on manifolds with boundary.

For instance, in [Per02] he introduced the functionals

$$\mathcal{F}(g,f) = \int_M (R + |\nabla f|^2) e^{-f} dvol, \qquad \lambda(g) = \inf_f \left\{ \mathcal{F}(g,f), \quad \int_M e^{-f} dvol = 1 \right\}$$

and scale invariant versions of them, the entropy functionals $\mathcal{W}$ and μ. They are monotone increasing along Ricci flow and are essential to prove the fundamental no local collapsing property for the Ricci flow.

Lott in [Lot12] supplements the $\mathcal{F}$ functional with a boundary term and studies its variational properties, while Ecker in [Eck07] considers the $\mathcal{W}$ functional in domains in $\mathbb{R}^n$ with evolving boundary. In both cases, an interesting link with Hamilton's differential Harnack inequality for the mean curvature flow appears, in the form of the boundary term of the variations of these functionals.

On a manifold with boundary which evolves by Ricci flow, it would be useful to know whether Perelman's energy and entropy functionals, properly modified, enjoy similar monotonicity. Moreover, it is still an open question under what conditions a no local collapsing theorem holds on compact manifolds with boundary. Are there any examples where it fails?

On a different issue, the main tool which allows Perelman to construct the Ricci flow with surgery for three dimensional manifolds is the *canonical neighbourhood theorem*. It briefly says that when the curvature at some point of the manifold reaches a certain threshold, then the flow around that point is close to a region in a κ-solution.

The theorem relies heavily on the Hamilton-Ivey pinching for 3d Ricci flow and the no-local collapsing property of compact Ricci flow. Unfortunately, these ingredients are not available to us in the study of the Ricci flow on compact three-manifolds with boundary. Nevertheless, it may still be possible to prove such a theorem in special situations or under additional assumptions, given the following observations.

(1) The Hamilton-Ivey estimate can be localized, as it was shown in [CXZ13].
(2) Assuming a uniform bound on the second fundamental form of the boundary, the boundary (if non-empty) of any blow-up limit will be totally-geodesic. The double of this limit has non-negative curvature, as it is a complete ancient 3d Ricci flow on which Hamilton-Ivey applies.
(3) Perelman's monotonicity formulas remain true when the boundary is minimal and the conformal class preserved along the flow, provided the manifold has cohomogeneity one. In general, it would be interesting to investigate whether the canonical neighbourhood theorem remains valid *assuming* that the whole flow or the boundary is non-collapsed.

1.5.3. *Applications-links to other geometric problems.* In the long run, I would like to investigate possible applications and links between the evolutionary boundary value problems mentioned above and other problems in geometry.

For instance, a flow approach could be useful to understand probems related to immersions in spaces of constant curvature in dimension three. See [And12a] and [And12b] for an elliptic approach to this problem by Anderson. In particular, it would be interesting to apply Ricci flow in the study of minimal or constant mean curvature (CMC) immersions of surfaces in $S^3, \mathbb{R}^3$ or $\mathbb{H}(3)$, one of the oldest subjects in differential geometry. One could probably start the Ricci flow from a three dimensional manifold with minimal boundary and properly pinched curvature, and ask whether the normalized flow converges to an Einstein metric. This, in dimension three, would give rise to a minimal immersion of the boundary. In a similar spirit, Ricci flow could be used to obtain CMC immersions of surfaces in these spaces, by flowing a filling manifold to one with constant curvature. These applications are essentially linked to the stability questions considered in a previous section.

2. Singularity formation of Type I Ricci flows.

Another direction of my research concerns the understanding of the singularity formation of Type I singular Ricci flows. A Ricci flow $(M, g(t))_{t\in[0,T)}$ is Type I if there is a $C > 0$ such that $\sup_{M\times[0,T)}(T-t)|\operatorname{Rm}(g(t))|_{g(t)} < +\infty$. Understanding this special case could give some insight on what might be true regarding the singularity formation of general Ricci flows.

It was shown by Naber in [Nab10] that any blow-up sequence of the form $(M, \tau_i^{-1} g(\tau_i t + T), p)$, $\tau_i \to 0$, converges in the Cheeger-Gromov topology to a Ricci flow $(N, h(t), q)_{t\in(-\infty,0)}$ induced by a shrinking Ricci soliton (N, h_0, f). Namely

$$\mathrm{Ric}_{h_0} + \mathrm{Hess}_{h_0} f = \frac{h_0}{2}, \tag{2.1}$$

and $h(t) = -t\phi_t^* h_0$, where $\phi_t : N \to N$ is the flow of the vector field $-\frac{1}{t}\nabla^{h_0} f$ with initial condition $\phi_{-1} = \mathrm{id}_N$. Such limits will be called *tangent flows* at p and in general may depend on the sequence τ_i.

We may define the singular set of the flow $(M, g(t))_{t\in[0,T)}$ as follows, following [EMT11].

Definition 1. *A point $p \in M$ is at the singular set Σ of $(M, g(t))_{t\in[0,T)}$ if there is no neighbourhood U of p such that*

$$\sup_{U\times[0,T)} |\,\mathrm{Rm}(g(t))|_{g(t)} < \infty. \tag{2.2}$$

Interestingly, it was shown by Enders, Müller and Topping in [EMT11], that only the points in Σ have non-trivial tangent flows (i.e. not flat).

2.1. **The size of the singular set.** The first direction of my research regarding the singularity formation of Type I flows is a stratification theorem for the singular set. Define the sets $\Sigma_0 \subseteq \Sigma_1 \subseteq ... \subseteq \Sigma_{n-2} \subseteq \Sigma$ as follows. For each $j = 0, ..., n-2$ let

$$\Sigma_j \;=\; \{x \in \Sigma,\ \text{no tangent flow at } x \text{ splits as } (N^{n-j-1}, h(t)) \times (\mathbb{R}^{j+1}, g_{Eucl})\}.$$

It turns out that the strata Σ_j differentiate in terms of the behaviour of their volume as the flow becomes singular. In the forthcoming paper [Gia15], after introducing a new notion of a density function $\Theta_g : M \to (0, 1]$ for Type I Ricci flows, which is lower semicontinuous with respect to smooth Cheeger-Gromov convergence, I obtain the following theorem. The core of the proof consists of a new splitting theorem for shrinking Ricci solitons and time dependent versions of standard dimension reduction arguments.

Theorem 6. *Fix $j = 0, \dots, n-2$ and let $s > j$. Then, there exist closed $A_i \subset \overline{\Sigma_j}$ ($i = 1, 2, \dots$), depending on j and s, such that $\Sigma_j \subset \bigcup_{i=1}^\infty A_i$ and*

$$\frac{Vol_{g(-\tau+T)}(A_i)}{\tau^{\frac{n-s}{2}}} \;\le\; C(j, s, i)\tau^\beta, \tag{2.3}$$

for some $\beta = \beta(s - j) \in (0, 1)$.

Moreover, for each $x \in \Sigma_0$ there exist $R_0, \bar\tau > 0$ such that

$$B_g(x, -\bar\tau + T, R_0\sqrt{\bar\tau}) \cap \{y \in M,\ \Theta_g(y) \le \Theta_g(x)\} \subseteq B_g(x, -\tau + T, R_0\sqrt{\tau}), \tag{2.4}$$

for every $\tau \in (0, \bar\tau]$. Here $B_g(x, t, r)$ denotes the metric ball of radius r centered at x with respect to the distance induced by $g(t)$.

Estimate (2.3) should be thought of as an analogue of the Hausdorff dimension estimates of the form $\dim \Sigma_j \le j$ which are valid in many other contexts, such as [Alm83, Fed70, Sim93, Whi97, CC97]. The following corollary also holds.

Corollary 1. *If $\Sigma = \Sigma_j$ it follows that for every $\varepsilon > 0$ there exist closed $A_i \subseteq \Sigma$, $i = 1, 2, ...,$ such that $\Sigma = \bigcup_{i=1}^\infty A_i$ and*

$$Vol_{g(-\tau+T)}(A_i) \le C(i, \varepsilon)\tau^{\frac{n-j}{2}-\varepsilon}, \tag{2.5}$$

for every $\tau \in (0,T]$. In particular we distinguish the following cases.

(1) *In general $\Sigma = \Sigma_{n-2}$ and $Vol_{g(-\tau+T)}(A_i) \leq C(i,\varepsilon)\tau^{1-\varepsilon}$.*

(2) *Suppose that the Weyl curvature satisfies*

$$\sup_{M\times[0,T)} |W_g|_g < \infty.$$

Then $\Sigma = \Sigma_1$ and $Vol_{g(-\tau+T)}(A_i) \leq C(i,\varepsilon)\tau^{\frac{n-1}{2}-\varepsilon}$.

Moreover, in both cases, for every $\delta > 0$ there is i_0 such that $Vol_{g(-\tau+T)}(\Sigma \setminus A_i) < \delta$ for every $i \geq i_0$ and $\tau \in (0,T]$.

2.2. **Directions for future research.**

2.2.1. *Quantitative stratification of a Type I Ricci flow.* A part of Theorem 6 which is unsatisfactory is that the volume estimates concern the sets A_i instead of the full strata Σ_j. This is because the coverings of Σ_j constructed in the proof require balls of different radii, depending on the scale on which the flow around each point looks self-similar. An improvement would thus be analogous to estimates on the Minkowski content of Σ_j.

Such estimates where obtained in [CN13] for the so called quantitative stratification of limits of sequences of Riemannian manifolds with lower Ricci curvature and volume bounds. Moreover, these estimates coupled with ε-regularity theorems produce new L^p estimates for the curvature. The techniques introduced are quite general and have found applications in a variety of other contexts.

Thus, motivated by [CN13] one may attempt to define a quantitative stratification $\mathcal{S}^j_{\eta,\tau}$ (where $\eta > 0$, $\tau \in (0,T]$ and $j = 0,...,n-2$) for a Type I Ricci flow $(M,g(t))_{t\in[0,T)}$, $T > 1$, as follows.

$$\begin{aligned}\mathcal{S}^j_{\eta,\tau} \;\; = \;\; & \{y \in M,\ \mathrm{dist}(B_g(y,-\tau'+T,R_0\sqrt{\tau'}), B_h(q,-\tau',R_0\sqrt{\tau'})) \geq \eta\sqrt{\tau'}, \\ & \text{for all shrinking solitons } (N,h(t),q)_{t\in(-\infty,0)} = (N',h'(t),q') \times (R^{j+1}, g_{Eucl}, 0) \\ & \text{and all } 0 < \tau \leq \tau' \leq 1\},\end{aligned}$$

for an appropriate notion of distance $\mathrm{dist}(A,B)$. It then seems natural to expect an estimate of the form

$$\mathrm{Vol}_{g(-\tau+T)}(T_{\sqrt{\tau}}\mathcal{S}^j_{\eta,\tau} \cap B_g(x,-1,R_0)) \leq C_\eta \tau^{\frac{n-j-\eta}{2}}, \tag{2.6}$$

to be true. Here, T_rA denotes the r-tubular neighbourhood of the set A at time $-r^2+T$. Moreover, as in [CN13], I expect that such estimates coupled with an appropriate ε-regularity theorem will provide new L^p curvature estimates for the Ricci flow.

2.2.2. *Uniqueness of tangent flows.* As was mentioned above, the tangent flow at a point $p \in M$ may depend on the particular sequence τ_i. A question which I would like to investigate is in which situations one can assert that the tangent flow is independent to the sequence of rescalings. Equivalently, does $\bar{g}(t) = e^t g(T - e^{-t})$, which evolves by

$$\frac{\partial}{\partial t}\bar{g} = -2\,\mathrm{Ric}(\bar{g}) + \bar{g}, \tag{2.7}$$

converge to a shrinking Ricci soliton as $t \to \infty$?

This turns out indeed to be true when a tangent flow is a compact shrinking Ricci soliton, [Ses06a, SW15], however the question is completely open in general. For instance, it is not known whether uniqueness holds when a tangent flow is the cylinder $S^{n-k}\times\mathbb{R}^k$ or an asymptotically conical shrinking Ricci soliton, although both scenaria arise as tangent flows of compact Ricci flows (see [AK04, Máx14]).

This question has a long history of investigation in the world of geometric PDE, going back to the work of Allard and Almgren [AA81] and Simon [Sim83]. In particular, the ideas in [Sim83] have been applied to prove uniqueness of compact tangent flows for mean curvature flow [Sch14] as well as the Ricci flow. Unfortunately, the existing Lojasiewicz-Simon inequalities are not sufficient to deal with non-compact singularities, as the rescalings of the flow (either the Ricci flow or the mean curvature flow) do not converge to the tangent flows uniformly.

However, in a remarkable recent work [CM15], Colding and Minicozzi proved that for the mean curvature flow and for cylindrical singularities (i.e. a tangent flow is a generalized cylinder $S^{n-k}\times\mathbb{R}^k$) uniqueness holds. This improves previous work of Colding, Ilmanen and Minicozzi [CIM15] in which it was shown that if at some point a tangent flow is cylindrical, then they all are, but the axis of the cylinder may not be unique.

The mean curvature flow and Ricci flow, although quite different in many aspects in their analysis, share many similarities. Jointly with Reto Müller, I intend to explore whether the ideas in [CIM15, CM15] can help provide a better understanding on the issue of uniqueness of cylindrical tangent flows for the Ricci flow.

3. Flowing spaces with conical singularities by Ricci flow.

A different direction of my research is on existence and uniqueness questions for the Ricci flow with non-smooth initial data. A motivation for this investigation arises from the problem of constructing a Ricci flow through singularities, since one could imagine restarting the flow once it becomes singular. Although for the mean curvature flow weak notions of solutions which allow it to continue past the first singular time are already known to exist (see for instance [Bra78, ES91, CGG91]), the analogous question for the Ricci flow remains open. The only exceptions are situations in which the behaviour of Ricci flow is understood well enough to perform surgery. Such are the remarkable works of Perelman [Per02, Per03b, Per03a] on three-dimensional and of Hamilton in [Ham97], and Chen and Zhu in [CZ06] on four-dimensional Ricci flow with positive isotropic curvature. Finally, singular three-dimensional Ricci flows through singularities have recently been constructed by Kleiner and Lott in [KL14], by taking limits of Ricci flows with surgery as the surgery parameter goes to zero.

There has already been some very interesting literature on the Ricci flow with non-smooth initial data. For instance, the work of Simon in [Sim09, Sim12] and Richard [Ric15] shows that it is possible to construct smooth Ricci flows starting from Gromov-Hausdorff limits of sequences of Riemannian manifolds with various curvature and non-collapsing conditions. Also, Koch and Lamm in [KL12] prove existence of the Ricci-DeTurck flow in $\mathbb{R}^n$ assuming merely continuous initial data, provided they are close enough to the Euclidean metric. Moreover, asymptotically conical Ricci expanders provide models of smooth Ricci flows flowing out of metric cones (see for instance [Der14, Der15, SS13]).

3.1. Short-time existence.

In collaboration with Felix Schulze I have been investigating the possibility of using linearly stable, asymptotically conical expanders to construct Ricci flows from initial data with isolated conical singularities. Here it is important to point out that recent results in [MW14] on the conical structure of shrinking Ricci solitons at infinity (those with Ricci curvature decaying to zero at infinity), give strong evidence that conical behaviour at the singular time, as in [Máx14], is in fact a quite general behaviour for the Ricci flow.

We say that a Riemannian manifold (M, g_0) has a conical singularity modeled at a cone $(C(X), g_c = dr^2 + r^2 g_X)$, where X is a compact Riemannian manifold, if there exists a C^∞ map $\phi : (0, r_0) \times X \to M$, diffeomorphism onto its image, such that

(1) ϕ extends continuously to $[0, r_0) \times X$ and $\phi(0, x) = p$.
(2) There exists a positive function $f(r)$, $r > 0$, with $\lim_{r\to 0} f(r) = 0$, such that

$$|\phi^* g_0 - g_c|_{g_c} + r|\nabla_{g_c}\phi^* g_0|_{g_c} + r^2|\nabla^2_{g_c}\phi^* g_0|_{g_c} < f(r),$$

A gradient Ricci expander (N, g_e, f) is a Riemannian manifold which satisfies $\mathrm{Ric}_{g_e} + \mathrm{Hess}_{g_e} f = -\frac{g_e}{2}$. If ϕ_t is the flow of the vector field $\frac{1}{t}\nabla^{g_e} f$ satisfying $\phi_1 = \mathrm{id}_N$, then $g_e(t) = t\phi_t^* g_e$ is a Ricci flow for $t > 0$. Moreover, in the case of an asymptotically conical expander, the flow $g_e(t)$ converges in the pointed Gromov-Hausdorff topology to the cone at infinity as $t \to 0$.

An expander is called strictly linearly stable if there exists a $\lambda > 0$ such that every compactly supported symmetric 2-tensor h on N satisfies

$$\int_N (|\nabla_{g_e} h|^2_{g_e} - 2\,\mathrm{Rm}(g^e)(h, h))e^f d\mu_{g_e} \geq \lambda \int_M |h|^2_{g_e} e^f d\mu_{g_e}, \tag{3.1}$$

where $\mathrm{Rm}(g_e)(h, h) = R^{g_e}_{ijkl} h_{il} h_{jk}$. The linear and dynamical stability of Ricci expanders has been studied by Deruelle in [Der14] and Jablonski, Petersen and Williams in [JPW14]. For instance, it is shown in [Der14] that the rotationaly symmetric, expanding gradient Ricci solitons, asymptotic to $(C(S^n), dr^2 + (cr)^2 g_{S^n})$ for $c \in (0, 1)$, which were constructed by Bryant are stable.

In a forthcoming paper we prove the following result.

Theorem 7. *Let (M, g_0) be a Riemannian manifold with a conical singularity at $p \in M$ modeled at a cone $C(X)$. Suppose there exists a gradient Ricci expander (N, h, f) asymptotic to $C(X)$ which is also strictly linearly stable. Then, there exists $T > 0$ and a smooth Ricci flow $(M', g(t))_{t\in(0,T]}$ with the properties*

(1) *The manifold M' results from (an appropriate) gluing of the manifolds M and N.*
(2) *$(M' g(t)) \to (M, g_0)$ as $t \to 0$ in the Gromov-Hausdorff sense, and smoothly away from p.*
(3) *$\sup_{M'} |\mathrm{Rm}(g(t))|_{g(t)} \leq \frac{C}{t}$, for some $C > 0$ and all $t \in (0, T]$.*

The main idea is to desingularize the conical singularity of (M, g_0) by constructing a family of smooth Riemannian manifolds $(M', g^0_s)_{s\in(0,s_0]}$ such that (M', g^0_s) converge to (M, g_0) as $s \to 0$. We do this by gluing $(N, g_e(s))$, the expander at scale s, onto (M, g_0). The difficulty then is to show uniform estimates for the induced Ricci flows, which then allows us to pass to the limit as $s \to 0$ to obtain a flow coming out of the singular initial data. Note that the curvature of (M', g^0_s) is blowing up, and the standard existence theory only guarantees existence and uniform estimates for the Ricci flow for a time interval of length proportional to s. This is the point that the stability of the expander plays a crucial role, in combination with Perelman's pseudolocality theorem. This scheme has previously been employed by Ilmanen, Neves and Schulze in [INS14], and Begley and Moore in [BM15] to construct solutions from non-regular initial data of the network flow and the Lagrangian mean curvature flow respectively. The Ricci flow case, however, poses additional difficulties in that the stability results for expanders concern the closely related Ricci-DeTurck flow instead.

3.2. **Directions for future research.** A natural question that follows from an existence result like Theorem 7 is that of uniqueness. One could try posing this question in the following way.

Given (M, g_0) a Riemannian manifold with an isolated conical singularity at $p \in M$, what is the widest class of smooth Ricci flows $(M', g(t))_{t\in(0,T]}$ so that the assumption $(M', g(t), q) \to (M, g_0, p)$ as $t \to 0$ determines the flow uniquely. Of course it might be that there are flows with different topology M', or it might be that there are different expanding Ricci solitons coming out of the same cone giving rise to different Ricci flows.

In the future I intend to explore these questions further, jointly with Felix Schulze and Miles Simon.

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